J. Fluid Mech. (1971), *vol.* 50, *part* 3, *pp.* 481–491
Printed in Great Britain

Irrotational flow past bodies close to a plane surface

By E. O. TUCK
Department of Applied Mathematics, University of Adelaide

(Received 3 August 1971)

A theroetical analysis is given for potential flow over, around and under a vehicle of general shape moving close to a plane ground surface. Solutions are given both in the form of a small-gap asymptotic expansion and a direct numerical computation, with close agreement between the two for two-dimensional flows with and without circulation. Some results for three-dimensional bodies are discussed.

1. Introduction

In this paper we consider steady irrotational flow of an inviscid incompressible fluid over, around and (especially) under a body of general shape situated close to a plane surface. The aim is to model some aspects of the aerodynamics of a vehicle moving over the ground with a very small clearance. Classical theoretical work on such ground effects (e.g. Milne-Thomson 1960, p. 506) requires clearances which are large compared with typical body dimensions.

The present analysis had its origins in investigations of ship and submarine hydrodynamics. For instance, Havelock (1939), Tuck (1966) and Tuck & Taylor (1970) have considered the so-called 'sinkage' problem for ships, in which to a certain degree of approximation one represents the free surface by a fixed plane and then computes vertical forces. These are, of course, situations of zero clearance. Newman (1965) solved the problem of a slender body of revolution moving close to a plane boundary, the clearance considered being of the same order of magnitude as the body width, and computed the force attracting the body towards the wall.

Shallow-water ship hydrodynamic problems in which flow takes place through a small gap between the ship's bottom and the sea floor were considered by Newman (1970), Tuck & Taylor (1970) and Taylor (1971). The concept of a local highly constrained flow through the gap matching an outer flow was also discussed by Widnall & Barrows (1970) for wing-like ground vehicles.

We approach the problem here from two different points of view. In the first place we present a small-gap theory, in which the solution is obtained formally (as in the work of Widnall & Barrows 1970) in the form of an asymptotic expansion with respect to the small parameter ϵ = gap size/typical body dimension. We develop this expansion in detail for a semicircular cylindrical vehicle and indicate how it can be generalized to an arbitrary body.

The second approach is a purely numerical treatment of the two-dimensional case in the manner of Giesing & Smith (1967). We now assume that the gap size is comparable with but not necessarily smaller than the body dimensions, and the accuracy of the results is limited only by the fineness of the mesh of points into which we divide the body profile. The resulting program is used to verify the small-gap theory for the semicircular case and, in addition, to provide flow computations for an automobile profile.

In the two-dimensional case an inevitable consequence of the assumption that the body is of general (non-wing-like) form is that we must be content with a non-unique solution within irrotational theory. That is, we lack a trailing-edge condition such as the Kutta condition which, in classical aerodynamics, prescribes for us the circulation around the body.

In fact we present here, both by small-gap asymptotic analysis and by numerical computation, the most general solution for this type of two-dimensional flow, namely a linear combination of a streaming flow without circulation and a flow due solely to a fictitious vortex located inside the body.

The choice of the correct value of the circulation is very difficult to make without recourse to viscous fluid considerations and indeed one may have real doubts about the legitimacy of any inviscid approach to bluff-body aerodynamics. However, there is reason to believe that some progress can be made using the present approach and that the solution given can, for a suitable choice of the circulation, represent to a reasonable approximation the actual flow over and under a profile such as that of an automobile.

The correct choice of the circulation is of course of critical importance if we are interested in the lift force on the vehicle. One point to realize, however, is that because of the presence of the ground plane this force is not necessarily zero when the circulation is zero. Indeed, it is not difficult to see that at zero circulation the net lift force is always *downward*, or toward the ground, because of the Venturi effect of flow through the gap.

In practice automobiles are more often than not subject to a positive (away from ground) lift force, indicating presence of circulation in such a direction as to slow down this under-vehicle flow. However, negative lift is not unheard of and is, for example, achievable by suitable design features (e.g. Marcell & Romberg 1970), so that one cannot dismiss the possibility of a zero or near to zero net circulation, which may be desirable from a number of points of view.

Although two-dimensional flow would seem to be a reasonable approximation for the centre profiles of most conventional automobiles, this is clearly unsatisfactory near the sides of automobiles and for other vehicles such as trains. We show how the small-gap theory may, at least to leading order, be carried out easily for three-dimensional bodies of arbitrary shape. The results have been worked out in detail for bodies whose upper surface consists of half an ellipsoid or a slender body of revolution.

2. Small-gap theory for circular cylinders

We first suppose that the vehicle is the nearly semicircular cylinder

$$r = (x^2+y^2)^{\frac{1}{2}} < a, \quad y > \epsilon a$$

whose flat bottom surface $y = \epsilon a$ is situated close to the ground plane $y = 0$, i.e. $\epsilon \ll 1$, as in figure 1. The problem to be solved involves finding a velocity potential ϕ which satisfies

$$\nabla^2\phi = 0 \tag{2.1}$$

exterior to the vehicle,
$$\phi \to Ux \quad \text{as} \quad r \to \infty, \tag{2.2}$$

$$\partial\phi/\partial r = 0 \quad \text{on} \quad r = a, \tag{2.3}$$

$$\partial\phi/\partial y = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad y = \epsilon a, \tag{2.4}$$

and
$$[\phi] = \kappa, \tag{2.5}$$

where $[\phi]$ is the net change in ϕ over any complete circuit surrounding the vehicle. The free-stream speed U and the circulation κ are both supposed given.

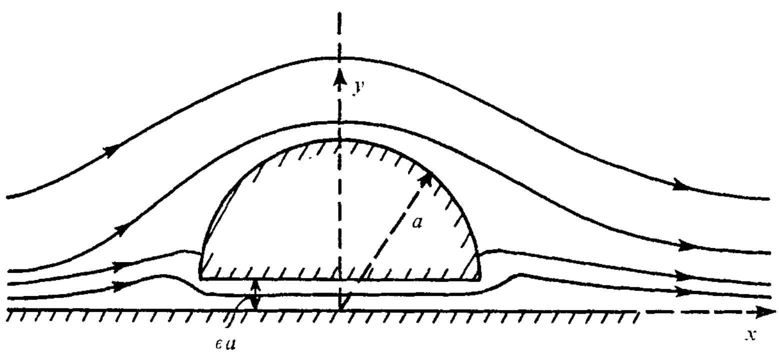

FIGURE 1. Sketch of flow situation for semicircular vehicle.

The asymptotic solution of the above boundary-value problem for small ϵ can be obtained in a systematic manner by the method of matched asymptotic expansions (Van Dyke 1964). Only the barest outline of the required analysis will be presented here, reliance being placed on the intuitive nature of the inner and outer expansions.

In the first instance it is clear that as $\epsilon \to 0$ the influence of the gap on the 'outer' flow over the top of the semicircle disappears. That is, the 'zero-order' solution for ϕ is nothing more than that for flow over the complete circle $r = a$, namely

$$\phi \to Ux(1+a^2/r^2) = \phi_0. \tag{2.6}$$

This is expected to be a good approximation everywhere except near and in the gap. Notice that this approximation does not yet take any account of the circulation κ, as is inevitable from the fact that circulation can only be present if the gap is present.

Since (2.6) is hardly satisfactory as a model of the flow with a gap present we must seek both an improved outer approximation and a different approximation which we can use in the gap itself, i.e. for $y = O(\epsilon a)$. In the outer region we introduce a sink at $(-a, 0)$ and a source at $(a, 0)$, each of strength ϵm, as a first-order correction to ϕ_0, modifying (2.6) to give

$$\phi \to \phi_0 + (\epsilon m/2\pi)\log(R/R') = \phi_1, \tag{2.7}$$

where
$$R^2 = (x-a)^2 + y^2 \tag{2.8}$$

and
$$R'^2 = (x+a)^2 + y^2. \tag{2.9}$$

The approximation ϕ_1 satisfies (2.1), (2.2) and (2.3), and clearly models the drawing off of fluid at the leading edge and return at the trailing edge. The source strength ϵm is to be determined. In the subsequent discussion we shall need the behaviour of ϕ_1 near the edges, e.g. as $(x, y) \rightarrow (a, 0)$, we have

$$\phi_1 \rightarrow (\epsilon m/2\pi) \log R + 2Ua - (\epsilon m/2\pi) \log 2a. \tag{2.10}$$

In order to determine the magnitude of the source strength ϵm we must look in detail at the flow near the edges. Figure 2 shows the flow near the trailing edge

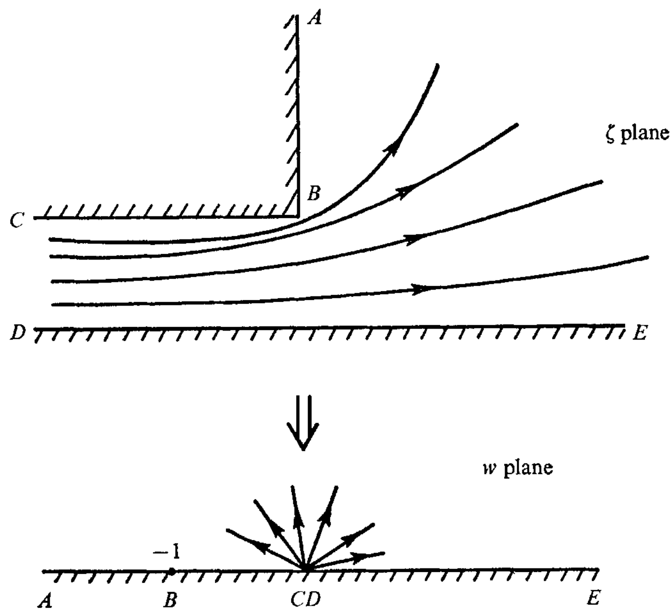

FIGURE 2. Flow in the trailing edge region, and its conformal mapping into the upper-half w plane.

$(a, 0)$, indicating a stream V emerging from the middle part of the gap as a source. The appropriate stretched co-ordinates in a 'blown-up' picture of the edge flow such as figure 2 are $\xi = (x-a)/\epsilon$ and $\eta = y/\epsilon$, these quantities remaining bounded as $\epsilon \rightarrow 0$.

The flow problem in the trailing-edge region may be solved immediately by the conformal mapping

$$(\pi/a)\,\zeta = 2(w+1)^{\frac{1}{2}} - 2\log\left((w+1)^{\frac{1}{2}}+1\right) + \log w, \tag{2.11}$$

which maps the flow region in the $\zeta = \xi + i\eta$ plane into the upper-half w plane, as indicated in figure 2. Clearly the appropriate solution in the w plane is a source at $w = 0$ and we obtain

$$\phi = (\epsilon m/4\pi) \log |w| + C, \tag{2.12}$$

which matches (2.10) as $w \rightarrow \infty$ if

$$C = (\epsilon m/2\pi) \log(\epsilon/\pi) + 2Ua. \tag{2.13}$$

The behaviour of the trailing-edge solution (2.12) in the gap region $w \to 0$, i.e. $(\pi/a)\zeta \to \log w + 2 - \log 2$, is

$$\begin{aligned}\phi &\to (\epsilon m/4\pi)((\pi/a)\zeta - 2 + 2\log 2) + C \\ &= Vx - \tfrac{1}{2}\kappa, \end{aligned} \tag{2.14}$$

where
$$V = m/4a \tag{2.15}$$

and
$$\begin{aligned}-\tfrac{1}{2}\kappa &= C - \frac{m}{4} - \frac{\epsilon m}{2\pi} + \frac{\epsilon m}{2\pi}\log 2 \\ &= \frac{\epsilon m}{2\pi}\left(-\frac{\pi}{2\epsilon} + \log\frac{2\epsilon}{\pi e}\right) + 2Ua. \end{aligned} \tag{2.16}$$

The significance of (2.14) is that it establishes the flow under the vehicle as that of a uniform stream V, where V and m are related by (2.15). The parameter m is itself determined from the circulation κ by means of (2.16). That κ as introduced in (2.14) is indeed the circulation prescribed by (2.5) is clear from the fact that our solution for ϕ is necessarily antisymmetric fore and aft. That is, the corresponding analysis for the *leading* edge would produce $\phi \to Vx + \frac{1}{2}\kappa$ instead of (2.14), the jump in ϕ at $x = 0$ being therefore of magnitude κ as required.

Combining (2.15) and (2.16) we have finally that the flow velocity V under the gap is given by

$$V = \frac{2U + \kappa/2a}{1 - (2\epsilon/\pi)\log(2\epsilon/\pi e)}. \tag{2.17}$$

The formula (2.17) is technically inconsistent as an asymptotic expansion with respect to ϵ and can be written as

$$V = 2U + \kappa/2a + O(\epsilon\log\epsilon), \tag{2.18}$$

in which the $O(\epsilon\log\epsilon)$ contribution can be obtained by expanding the denominator of (2.17). The error in (2.17) is at most $O(\epsilon^2\log^2\epsilon)$.

The zero-order result (2.18) is of some interest, indicating as it does that in the absence of circulation ($\kappa = 0$) the gap velocity is $V = 2U$, twice the stream velocity and the same as the maximum velocity of slip at the top of the vehicle. On the other hand, in the absence of a stream ($U = 0$) the velocity $V = \kappa/2a$ due to circulation alone is precisely the value needed to generate all the circulation κ from the base $2a$; thus circulation produces vanishingly small velocity components over the upper surface of the body. We also observe that clockwise circulation of amount $\kappa = -4Ua$ produces *zero* net flow ($V = 0$) through the gap. This is the value of κ we should obtain by use of a Kutta condition at the edges, leading to stagnation conditions there and hence, necessarily, stagnation conditions throughout the gap.

The actual gap size parameter ϵ affects the gap velocity V only through the $\epsilon\log\epsilon$ correction term to (2.18) and we can see from (2.17) that it has the effect of reducing the gap velocity. Thus at $\epsilon = 0{\cdot}05$ we obtain a 13 % reduction and at $\epsilon = 0{\cdot}02$ a 7 % reduction. As we shall see in the following section, formula (2.17) gives excellent agreement with 'exact' computations. However, even the zero-order formula (2.18) is a reasonable estimate and we indicate in § 4 how this zero-order result may be obtained for general body shapes.

It should be emphasized again that, although we have for the sake of brevity relied heavily upon intuitive arguments in this section, the asymptotic analysis can be and has been carried out using the formal apparatus of matched asymptotic expansions. Such a careful expansion was used by Widnall & Barrows (1970) for the case of their wing-like bodies.

3. Two-dimensional numerical solutions

Several numerical techniques exist for solving Laplace's equation subject to Neumann-type boundary conditions for arbitrary boundary geometries. Of special note are the 'source-distribution integral equation' techniques used by Hess & Smith (1964) for non-lifting bodies in three dimensions, by Giesing & Smith (1967) for two-dimensional lifting sections under a free surface, and by Frank (1967) for unsteady free-surface oscillations. Since these techniques are well known and the present case is very similar to that treated by Giesing & Smith (1967), we present here only the results of the computations.

The input of the program consists of a set of co-ordinates defining the shape of the body in the form of an approximating polygon. The number, length and location of the segments are chosen intuitively and by trial and error in order to produce a sufficient density of segments at parts of the body where the flow is varying rapidly. It is very difficult to give a satisfactory *a priori* criterion for this choice or even to be sure of the accuracy achieved once the choice has been made. However, about 60 segments have been found to be satisfactory, yielding results which appear to have errors of not more than 1 % where the flow is slowly varying and not more than 10 % where rapid variations take place. These error estimates are by comparison with similar 40 and 80 segment data. The computer time required for 60 segments is 6 sec (CDC 6400).

The initial output of the program is the velocity u of slip along the body contour, or rather the components u_S, u_V of u due separately to the stream U and the circulation κ. That is, we write

$$u = Uu_S + (\kappa/B)\,u_V, \tag{3.1}$$

where u_S is the slip velocity created by a stream of unit magnitude in the absence of circulation, whereas u_V is the slip velocity created in the absence of a stream by a fictitious unit vortex inside the body. The parameter B, taken as the base length of the body, is included to make u_V non-dimensional.

Other outputs readily available are the pressure coefficients, defined via Bernoulli's equation by

$$p = p_\infty + \tfrac{1}{2}\rho U^2[C_p^{SS} + (\kappa/UB)\,C_p^{SV} + (\kappa/UB)^2\,C_p^{VV}], \tag{3.2}$$

where p_∞ is the free-stream pressure and

$$C_p^{SS} = 1 - u_S^2, \tag{3.3}$$

$$C_p^{SV} = -2u_S u_V, \tag{3.4}$$

$$C_p^{VV} = -u_V^2 \tag{3.5}$$

are pressure coefficients due respectively to the stream alone, to coupling between stream and vortex and to the vortex alone. Finally we integrate each of the pressure coefficients with respect to x to obtain the upward lift force per unit span, namely

$$L = \tfrac{1}{2}\rho U^2 B[C_L^{SS} + (\kappa/UB)\, C_L^{SV} + (\kappa/UB)^2\, C_L^{VV}]. \tag{3.6}$$

Note that in the absence of a ground plane, the Kutta–Joukowski theorem demands that $C_L^{SS} = C_L^{VV} = 0$, and that $C_L^{SV} = -2$, whereas in the presence of the ground plane all three are non-zero.

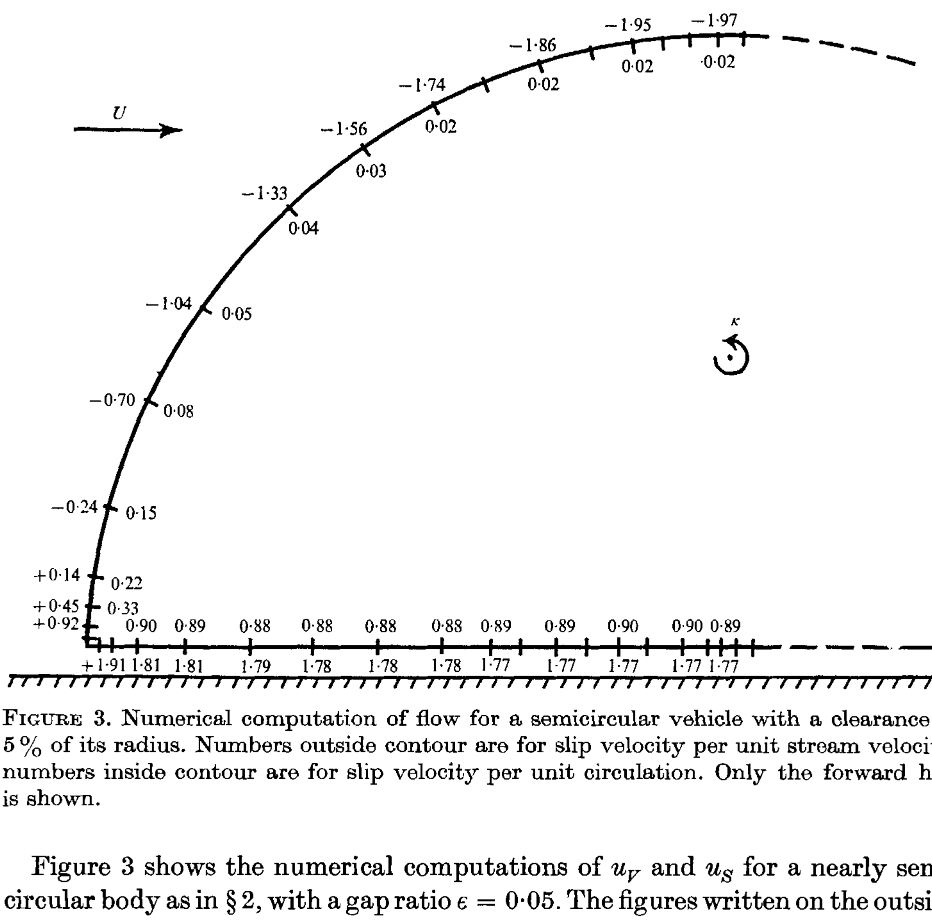

FIGURE 3. Numerical computation of flow for a semicircular vehicle with a clearance of 5% of its radius. Numbers outside contour are for slip velocity per unit stream velocity, numbers inside contour are for slip velocity per unit circulation. Only the forward half is shown.

Figure 3 shows the numerical computations of u_V and u_S for a nearly semicircular body as in § 2, with a gap ratio $\epsilon = 0{\cdot}05$. The figures written on the outside of the contour are those for u_S and those on the inside are those for u_V. Note that the under-vehicle flow is nearly uniform and the value 1·77 for u_S agrees closely with the value 1·75 given by the small-gap result (2.17). Note also that the above-vehicle values of u_V are very much smaller than the under-vehicle values, as is also expected from the small-gap theory. The above-vehicle values of u_S are well predicted by ϕ_1 as in (2.7), including the location of the stagnation points. The accuracy of the overall numerical process does not seem to be significantly affected by the mild local singularity at the right-angle corners.

In figure 4 we present similar computations of u_S and u_V for an automobile-like profile. No attempt has been made to model the rough undersurface of an actual automobile or to model details such as radiator grilles or bumper bars. Note again the relative constancy of u_S and u_V over this profile's flat bottom, but now with a smooth transition around the smooth 'edges' at front and rear towards stagnation points of u_S. Also, note again how small the values of u_V over the top of the vehicle are relative to u_S there and to u_V and u_S under the vehicle.

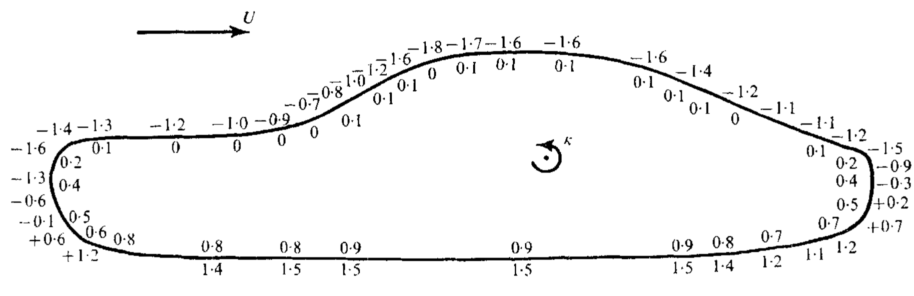

FIGURE 4. Numerical computation of two-dimensional flow for an automobile-like profile. Numbers outside contour are for slip velocity per unit stream velocity, numbers inside contour are for slip velocity per unit circulation.

In assessing the relevance of the results given in figure 4 to the real automobile situation this last property is of great significance. Of course we cannot maintain that the computed results have any significance at the extreme rear end of the vehicle, where separation and wake formation takes place. However, so long as the wake remains reasonably thin (as seems likely for a well-designed automobile, if not for a semicircle!) the main effect of this wake on the flow elsewhere will be in the determination of κ. But since the flow over the top of the vehicle is insensitive to κ we may expect that this flow is reasonably well predicted by u_S alone. Regrettably few wind-tunnel measurements on automobiles have appeared in the open literature; however, one may favourably compare figure 4 qualitatively with pressure plots given by Morelli (1964), Tetens (1966) or Potthoff (1969), using (3.3) to convert u_S to C_p^{SS}.

On the other hand, the correct choice of κ is clearly critical for the under-vehicle flow since the values of u_V and u_S are comparable there. This means that more information about the value of κ is needed before the under-vehicle flow can be determined by purely theoretical considerations. For example, we may consider that the rear stagnation point of u_S in figure 4 is a little too high. In order to shift this stagnation point downward even a small amount we need a quite significant *negative* circulation κ, which will choke off the under-vehicle flow and induce upward lift forces. This sensitivity to κ implies sensitivity to small design changes.

The lift coefficients for this particular vehicle are $C_L^{SS} = -0{\cdot}21$, $C_L^{SV} = -2{\cdot}53$ and $C_L^{VV} = -0{\cdot}69$. Thus at zero circulation we have a negative (downward) lift coefficient of $-0{\cdot}21$. As we introduce negative (clockwise) circulation the lift first becomes positive (upward) at $\kappa/UB = -0{\cdot}08$, a value which is not sufficient to depress the rear stagnation point to a noticeable extent in figure 4. The lift remains positive for $-3{\cdot}58 < \kappa/UB < -0{\cdot}08$, a range which includes the value $\kappa/UB = -1{\cdot}56$ at which the under-vehicle flow is reduced to zero.

In general very little can be said about drag in an inviscid theory such as this and we should certainly expect to have to provide an adequate theory for the wake to make significant progress. However, we may speculate that a zero value for the circulation κ would be desirable from the point of view of induced drag, since absence of circulation in a two-dimensional flow implies absence of trailing vortices when this flow is integrated across the vehicle. Such trailing vortices are a real and readily observable phenomenon for most automobiles (Potthoff 1969), and their elimination, if possible, would no doubt be desirable.

4. The small-gap theory for arbitrary bodies

A general three-dimensional extension of the theory in § 2, including circulation, is of course exceedingly difficult. Widnall & Barrows (1970) have made some progress with three-dimensional wing-like vehicles, for which the Kutta condition enables determination of the circulation at each spanwise position and hence the strength of the trailing vortices.

We provide here only the result for the case when such vortices are entirely absent, i.e. when the section-wise circulation is zero. In this case the small-gap theory may be generalized directly; instead of the source–link pair of § 2 we now have a distribution of sources and sinks around the 'intersection' curve C of the vehicle and the ground plane.

That is, if ϕ_0 again denotes the zero-order solution to the outer problem in the absence of a gap obtained by solving the 'double-body' problem formed by reflecting the vehicle in the ground plane $y = 0$, then we write for the first-order outer solution

$$\phi \to \phi_0 + \epsilon \int_C m(s)\, G(x,y,z; X(s), O, Z(s))\, ds = \phi_1. \tag{4.1}$$

Here $\epsilon m(s)$ is the source distribution, to be determined, while $G(x,y,z;X,Y,Z)$ is the Green's function for the double body, i.e. the potential for a unit source at the point (X, Y, Z), evaluated at the point (x,y,z) and satisfying $\partial G/\partial n = 0$ on the body. The curve C is parametrized by $x = X(s)$, $y = 0$, $z = Z(s)$, where s is arc length along C.

For complicated body shapes the function G may be difficult to determine, even numerically. Fortunately, for the zero-order inner velocity (cf. (2.18)) a knowledge of G is not required. This is because, to leading order in ϵ, we need only match the *tangential* velocity component along C with an inner under-vehicle flow. The normal component is then matched by the local source strength to give the first correction (cf. (2.17)) to the inner flow.

Thus, if Φ denotes the inner velocity potential, and the bottom of the vehicle is given by

$$y = \epsilon H(x,z), \tag{4.2}$$

it is not difficult to show by stretching the y co-ordinate that Φ must satisfy

$$\frac{\partial}{\partial x} H \frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial z} H \frac{\partial \Phi}{\partial z} = 0 \quad \text{inside} \quad C. \tag{4.3}$$

On matching the tangential velocity (i.e. the potential Φ itself) to the outer solution across C, we have

$$\Phi = \phi_0 \quad \text{on} \quad C. \tag{4.4}$$

The boundary-value problem to solve (4.3) subject to (4.4) is a classical interior Dirichlet problem. In the special case of a flat-bottomed vehicle, H = constant, the differential operator in (4.3) reduces to the Laplacian in the (x, z) plane. Once the potential Φ has been determined, the source strength $m(s)$ follows in terms of the derivative $\partial\Phi/\partial n$ normal to C; in fact we have

$$\partial\Phi/\partial n = m/4H, \tag{4.5}$$

since the flux $\epsilon H\, \partial\Phi/\partial n$ emerging at C from beneath the vehicle into the quarter plane $[y > 0, (x, z)$ outside $C]$ must equal one quarter of the flux ϵm of the local source at s = constant.

The above arguments are intuitively based but have been confirmed by detailed asymptotic analysis for the case of a hemispherical vehicle and for a vehicle whose upper surface is half of a slender body of revolution of general shape. One feature of the zero-order solution for the hemispherical vehicle, common to all semi-ellipsoidal vehicles with flat bottoms, is that the Dirichlet problem (4.3), (4.4) has the solution

$$\Phi = Vx \tag{4.6}$$

for some constant V. That is, to zero order in ϵ the under-vehicle flow for flat-bottomed semi-ellipsoidal vehicles is simply a uniform stream V. The boundary condition (4.4) implies that the value of V must equal the velocity of slip along the extreme side of the vehicle where C (an ellipse) is parallel to the x axis. This happens also to be the *maximum* above-vehicle flow velocity and, for example, $V = \frac{3}{2}U$ for spheres.

In the general case we cannot expect a uniform stream beneath the vehicle but the present model will still tend to produce a rather high velocity, comparable with the accelerated flows at the side or top of the vehicle. Such high velocities would probably be substantially reduced in practice by viscous effects.

REFERENCES

FRANK, W. 1967 Oscillation of cylinders in or below the free surface of deep fluids. *Naval Ship R. & D. Center, Washington Rep.* no. 2375.

GIESING, J. P. & SMITH, A. M. O. 1967 Potential flow about two-dimensional hydrofoils. *J. Fluid Mech.* **28**, 113–129.

HAVELOCK, T. H. 1939 Note on the sinkage of a ship at low speeds. *Z. angew. Math. Mech.* **19**, 202–205.

HESS, J. L. & SMITH, A. M. O. 1964 Calculation of non-lifting potential flow about arbitrary three-dimensional bodies. *J. Ship Res.* **8**, 22–44.

MARCELL, R. P. & ROMBERG, G. F. 1970 Daytona's wind tunnel proven design is confirmed by race wins. *Soc. Auto Engrs J.* **78**, 52–55.

MILNE-THOMSON, L. M. 1960 *Theoretical hydrodynamics*, 4th ed. Macmillan.

MORELLI, A. 1964 Theoretical method for determining the lift distribution on a vehicle. *Proc. Int. Auto. Tech. Cong.* (*F.I.S.I.T.A.*) *Tokyo*, 140–151.

NEWMAN, J. N. 1965 The force and moment on a slender body of revolution moving near a wall. *Naval Ship R. & D. Center, Washington Rep.* no. 2127.

NEWMAN, J. N. 1970 Lateral motion of a slender body between two parallel walls. *J. Fluid Mech.* **39**, 97–115.

POTTHOFF, J. 1969 Air resistance and lift of modern motor vehicles (in German). *First Symp. Road Vehicle Aero.* (ed. A. Scibor-Rylski). London: City University.

TAYLOR, P. J. 1971 Ph.D. thesis, University of Adelaide.

TETENS, T. A. 1966 Development of a body ventilation system. *Gen. Motors Eng. J.* **3** (1), 32–37.

TUCK, E. O. 1966 Shallow-water flows past slender bodies. *J. Fluid Mech.* **26**, 81–95.

TUCK, E. O. & TAYLOR, P. J. 1970 Shallow water problems in ship hydrodynamics. *Proc. Eighth Symp. on Naval Hydro., Pasadena.* Washington: Office of Naval Research.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.

WIDNALL, S. E. & BARROWS, T. M. 1970 An analytic solution for two- and three-dimensional wings in ground effect. *J. Fluid Mech.* **41**, 769–792.